

Siewert solutions of transcendental equations, generalized Lambert functions and physical applications

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Abstract

We review the exact solutions of several transcendental equations, obtained by Siewert and his co-workers, in the '70s. Some of them are expressed in terms of the generalized Lambert functions, recently studied by Mezö, Baricz and Mugnaini. For some others, precise analytical approximations are obtained. In two cases, the asymptotic form of Siewert's solutions are written as Wright ω functions.

1 Introduction

In a series of papers published between 1972 and 1976 [1] /S50, [2] /S52, [3] /S57, [4] /S53, [5] /S56, [6] /S59, [7] /S62, [8] /S63, [9] /S68, [10] /S71, [11] /S80, [12] /S89, [13] /S100, [14] /S108, Siewert and his co-workers - Burniston (for [1], [2], [4], [6], [7], [10], [11]), Phelps III (for [13], [14]), Essig (for [5]), Dogget (for [10]) and Burkart (for [8]) - studied the solutions of several transcendental equations, important for their physical applications. All the aforementioned publications are available, with open access, on Siewert's web page [15]; the symbols /S50, /S52, etc., in the first lines of this paragraphs, indicate the number of the respective paper in Siewert's publication list. The approach used in these papers is based "on complex variable analysis and requires ultimately a canonical solution of a certain Riemann problem; the solution of the suitably posed Riemann problem follows immediately from the work of Muskhelishvili [16]", as stated in [2]. The effort invested in this vast research is impressive, and the results are a pioneering and extremely valuable contribution to the development of the theory of transcendental equations. In the same time, the solutions obtained in this way are, in general, very complicated and difficult to use in practical physical applications.

Recently, the interest for these solutions increased, as some of them can be expressed in terms of generalized Lambert functions, and put in a much more

usable form, according to the results obtained by Mezö, Baricz [17] and Mugnaini [18]. The applications of the theory of generalized Lambert functions to various physical problems were presented in [19], [20] and [21].

From the point of view of applied physics, the efforts in getting approximate analytical solutions to the same transcendental equations produced, independently, useful results. The interference between the progress made in mathematical physics and in applied mathematics (or in simple theories of applied physics) was not discussed systematically, even if the subject seems quite interesting. It is the main goal of the present paper to fill this gap.

So, author's intention was to interconnect results obtained in areas with a small overlapping - mathematical physics, magnetism, quantum mechanics, polymer physics, astronomy, solar energy conversion. The central contribution of this paper is to point out to approximate solutions of Siewert's transcendental equations and, whenever possible, to obtain approximate expressions for generalized Lambert functions which describe these exact solutions.

The structure of this article is the following. In the second section, we shall discuss a transcendental equation involving the Langevin function. Its exact solution will be written in terms of a generalized Lambert function. Using an analytical approximation of the inverse Langevin function, recently proposed by Kröger, we find an approximate expression for this solution, with a relative error smaller than 5×10^{-3} . Such approximations are useful not only in para- or superparamagnetism, but also in polymer physics and in solar energy conversion.

In Section 3, we shortly discuss two equations involving hyperbolic and (linear) algebraic functions. The next one will be devoted to an equation involving trigonometric and hyperbolic functions. Using an algebraic approximation for the tan function, the solution of the transcendental equation is written as a $W(s; t; a)$ generalized Lambert function. An over-simplifying approximation of the hyperbolic function, of interest for applied physics, is also mentioned. In Section 5, the asymptotic solutions of two transcendental equations are expressed in terms of the Wright ω function. In Section 6, several equations involving the Lambert and generalized Lambert functions are mentioned, and in Section 7, transcendental equations involving trigonometric and (linear) algebraic functions are discussed. An approximate, quite precise solution of the Kepler equation for elliptic orbits is discussed in detail. Section 8 is devoted to conclusions.

2 The Langevin function and its inverse

In [11], Siewert and Burniston obtain an exact analytical solution of the equation:

$$x \coth x = \alpha x^2 + 1 \quad (1)$$

It can be written in terms of the Langevin function

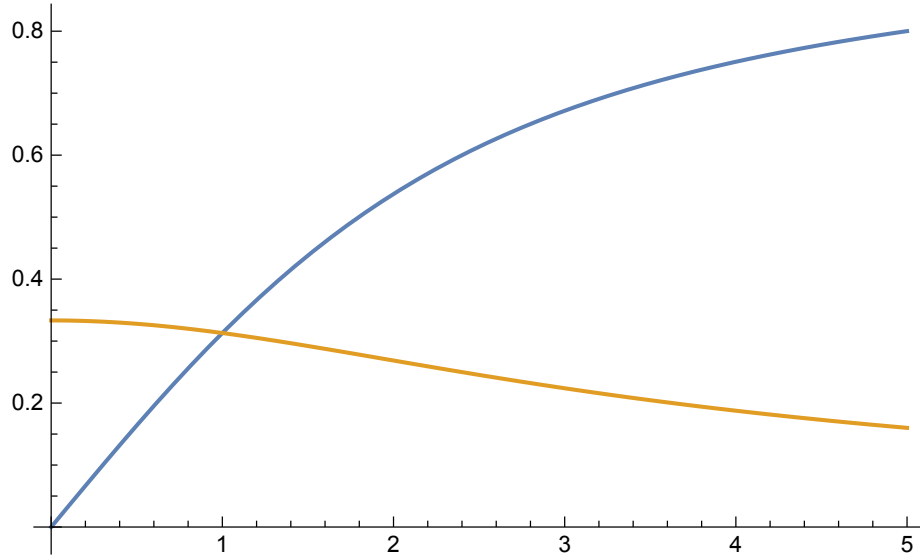


Figure 1: The plots of $L(x)$ (black) and $L(x)/x$ (red).

$$L(x) = \coth x - \frac{1}{x} \quad (2)$$

as:

$$L(x) = \alpha x \quad (3)$$

It is easy to see that $L(x)$ is an odd, and $L(x)/x$ - an even function of x .

In order to express the solution $x(\alpha)$ of this equation in terms of generalized Lambert functions, we shall put it in the form:

$$e^{2x} \frac{(\alpha x^2 - x + 1)}{(\alpha x^2 + x + 1)} = 1 \quad (4)$$

As:

$$\alpha x^2 - x + 1 = \alpha (x - x_{1\alpha})(x - x_{2\alpha}) \quad (5)$$

$$\alpha x^2 + x + 1 = \alpha (x + x_{1\alpha})(x + x_{2\alpha}) \quad (6)$$

with:

$$x_{1\alpha} = \frac{1 + \sqrt{1 - 4\alpha}}{2\alpha}, \quad x_{2\alpha} = \frac{1 - \sqrt{1 - 4\alpha}}{2\alpha} \quad (7)$$

we have:

$$e^{2x} \frac{(2x - 2x_{1\alpha})(2x - 2x_{2\alpha})}{(2x + 2x_{1\alpha})(2x + 2x_{2\alpha})} = 1 \quad (8)$$

so (4) becomes:

$$e^{2x} \frac{(2x - t_1)(2x - t_2)}{(2x - s_1)(2x - s_2)} = 1 \quad (9)$$

with:

$$t_1 = 2x_{1\alpha}, \quad t_2 = 2x_{2\alpha}, \quad s_1 = -2x_{1\alpha} = -t_1, \quad s_2 = -2x_{2\alpha} = -t_2 \quad (10)$$

and its solution can be written as a generalized Lambert function:

$$x(\alpha) = \frac{1}{2} W(2x_{1\alpha}, 2x_{2\alpha}; -2x_{1\alpha}, -2x_{2\alpha}; 1) \quad (11)$$

It seems that the value $\alpha = 1/4$ plays no special role in the aspect of the function $x(\alpha)$, even if the parameters t_1 , t_2 are real for $\alpha < 1/4$ and complex for $\alpha > 1/4$.

The Langevin function has been firstly introduced in the context of classical theory of paramagnetism, where it gives the magnetization M as a function of the external magnetic H field and temperature T :

$$M = n\mu L\left(\frac{\mu H}{k_B T}\right) \quad (12)$$

(see for instance [22], eq. (9.2)). This can be considered the equation of state for a classical paramagnet. The same formula is valid for superparamagnetic nanoparticles, at high enough values of temperature T [23], [24].

The Langevin function is a particular case of the Brillouin function B_S , defined as:

$$B_S(x) = \frac{2S+1}{2S} \coth\left(\frac{2S+1}{2S}x\right) - \frac{1}{2S} \coth\left(\frac{1}{2S}x\right) \quad (13)$$

Indeed,

$$B_\infty(x) = L(x) \quad (14)$$

It is easy to see that, if $0 < x < \infty$, then:

$$0 < L(x) < 1 \quad (15)$$

and:

$$0 < \frac{L(x)}{x} < \frac{1}{3} \quad (16)$$

The Langevin function and its inverse are relevant not only for magnetism, but also for other domains of physics with important practical applications,

as polymers (polymer deformation and flow) [25], [26], [27], [28] or solar energy conversion (daily clearness index) [30], [29]. Researchers in these fields proposed a large number of useful analytical approximations for $L(x)$ and $L^{-1}(x)$. Less precise algebraic approximations for $B_S(x)$ and $B_S^{-1}(x)$, but of real pedagogical interest, have been also obtained by Arrott [31]. We shall exemplify the usefulness of such formulas in the context of eq. (3).

Taking the inverse Langevin function in both sides of (3), we get:

$$L^{-1}(L(x)) = L^{-1}(\alpha x) = x \quad (17)$$

Let us use, for $L^{-1}(x)$, the very simple and precise approximation proposed by Kröger, see eq. (10) of [26]:

$$L^{-1}(x) = \frac{3x}{(1-x^2)(1+0.5x^2)} \quad (18)$$

In this case, the transcendental equation

$$L^{-1}(\alpha x) = x \quad (19)$$

gives an approximate, but simple algebraic equation, whose physically convenient root is:

$$x(\alpha) = \frac{1}{\alpha} \sqrt{\frac{\sqrt{3(3-8\alpha)} - 1}{2}} \quad (20)$$

The identity

$$f(\alpha) = \frac{x(\alpha) \coth x(\alpha)}{\alpha x(\alpha)^2 + 1} = 1 \quad (21)$$

where $x(\alpha)$ is replaced with the approximate solution (20), is fulfilled with a relative error less than 0.003, as we can see in the plot of Fig.2.

So, we have the approximate relation:

$$W\left(\frac{1+\sqrt{1-4\alpha}}{\alpha}, \frac{1-\sqrt{1-4\alpha}}{\alpha}; -\frac{1+\sqrt{1-4\alpha}}{\alpha}, -\frac{1-\sqrt{1-4\alpha}}{\alpha}; 1\right) \simeq \frac{1}{\alpha} \sqrt{2\left(\sqrt{3(3-8\alpha)} - 1\right)} \quad (22)$$

The algebraic approximations for the Brillouin functions B_S , proposed by Arrott [31], are not very precise, but however very useful for pedagogical purposes. The approach outlined in [26] might produce much better results.

If we use Cohen's approximation, eq. (F3) of [26]:

$$L^{-1}(y) = \frac{y\sqrt{3-y^2}}{1-y^2} \quad (23)$$

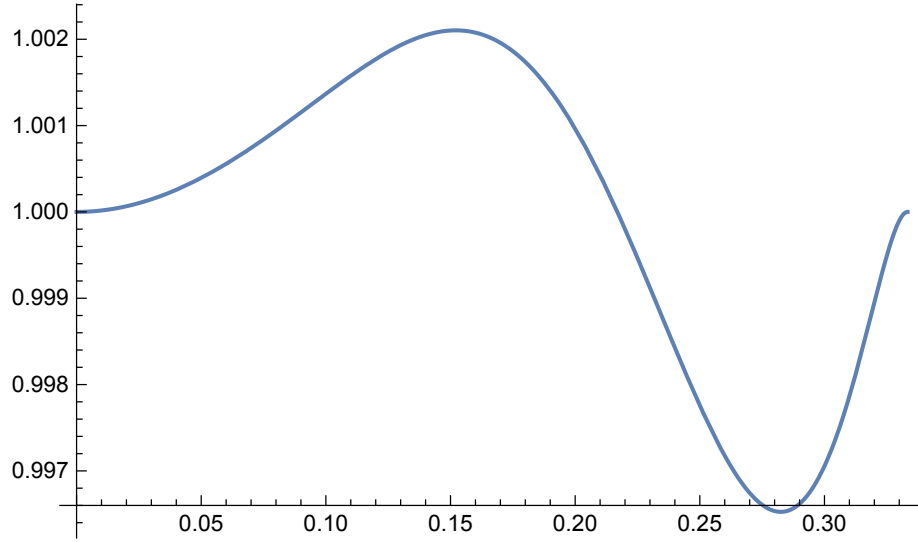


Figure 2: The plots of $f(a)$, eq. (21) and of the constant function $y = 1$.

we obtain, following the same steps

$$y(\alpha) = \frac{1}{\sqrt{2}\alpha} \sqrt{\alpha\sqrt{\alpha^2 + 8} - \alpha^2 + 2} \quad (24)$$

which satisfies the identity

$$g(\alpha) = \frac{y(\alpha) \coth y(\alpha)}{\alpha y(\alpha)^2 + 1} = 1 \quad (25)$$

- where $y(\alpha)$ is defined by (25) - with a much larger error compared to (21), as we can see in Fig. 3.

3 Equations involving hyperbolic and algebraic functions

In [8], the authors study the double zeros of the equation:

$$x = \tanh(ax + b) \quad (26)$$

If $b = 0$, one obtains a numerical value for a , namely $a = 1$, equivalent to the determination of the Curie temperature (see for instance [32], eq. (6.15) or [33], Ch. 15, eq. (8)).

Geometrically, a double zero of (26) means that the line $y_1(x) = x$ is tangent to the curve $y_2(x) = \tanh(ax + b)$, consequently:

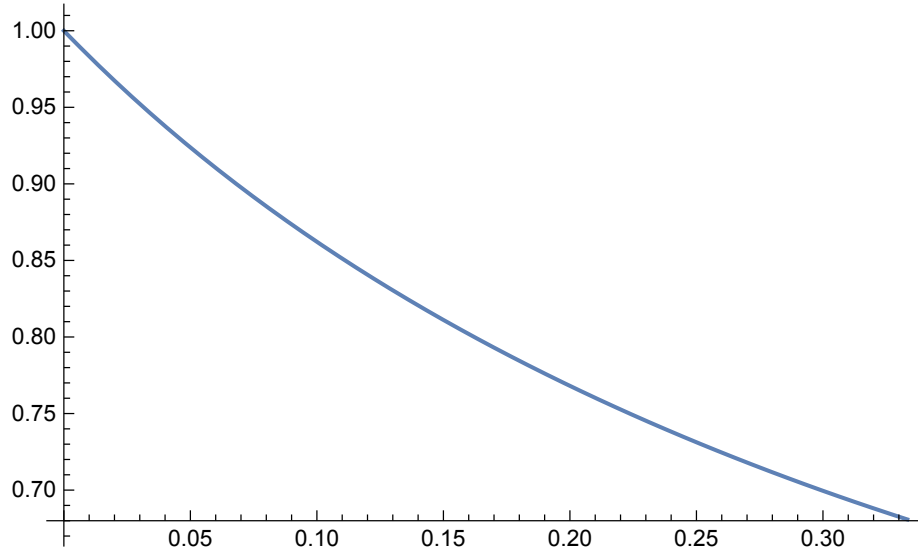


Figure 3: The plots of $g(a)$, eq. (25), and of the constant function $y = 0.7$.

$$\frac{d}{dx} \tanh(ax + b) = -a (\tanh^2(b + ax) - 1) = 1$$

But:

$$\tanh(ax + b) = x$$

so the tangency condition becomes:

$$a(1 - x^2) = 1 \rightarrow a = \frac{1}{1 - x^2}$$

and the solution of the problem is reduced to solving eq. (2) of [8], which can be written as:

$$x = \tanh\left(\frac{x}{1 - x^2} + b\right) = B_{1/2}\left(\frac{x}{1 - x^2} + b\right) \quad (27)$$

We can see that, actually, the problem involves only one parameter, b .

The inverse of \tanh can be expressed in terms of \ln , but it does not produce a simpler equation. A precise algebraic approximation of these functions could be of interest, as discussed at the end of the previous section.

In [5], Siewert and Essig solve the Weiss equation of ferromagnetism:

$$\zeta = \tanh \frac{1}{2} (jz\zeta + h) \quad (28)$$

Alternative ways of solving, exactly or approximately, this equation were presented in [34], where the exact solution is written in terms of a Lambert generalized function. The solution for the case $h = 0$ was written as a generalized Lambert function in [19], [20].

4 Equations involving trigonometric and exponential functions

In [13] and [14], the authors obtain the solutions of an equation basic to the theory of vibrating plates:

$$a \tan x + \tanh x = 0 \quad (29)$$

which appears also in quantum mechanics and electromagnetism.

We can make a certain progress in finding an approximate analytic solution of this equation using the algebraic approximation for $\tan x$ [35]:

$$\tan x \simeq \frac{0.45x}{1 - \frac{2}{\pi}x}, \quad 0 < x < \frac{\pi}{2} \quad (30)$$

This formula can be easily extended for any real x [36]. Replacing $\tan x$ in (29) according to (30), we get:

$$e^{2x} \frac{2x - \frac{\pi}{(1-0.45a\frac{\pi}{2})}}{2x - \frac{\pi}{(1+0.45a\frac{\pi}{2})}} = 1 \quad (31)$$

$$x(a) = \frac{1}{2}W\left(\frac{\pi}{(1-0.45a\frac{\pi}{2})}; \frac{\pi}{(1-0.45a\frac{\pi}{2})}; 1\right) \quad (32)$$

Eq. (31) is quite similar to the equation satisfied by the inverse Langevin function:

$$e^{2x} = \frac{A+1}{A-1} \frac{x + \frac{1}{A+1}}{x + \frac{1}{A-1}} \quad (33)$$

so the recipes for obtaining L^{-1} could be useful also for an approximate evaluation of W in (32).

As $\tanh x$ is a very slowly varying function, precise approximate solutions for the n -th root ($n > 1$) of (29) can be obtained as follows. Let us consider, for an illustrative example, that $a = -1$.

For the first root larger than 2π , one can approximate $\tanh x$ as:

$$\tanh x \simeq \tanh\left(2\pi + \frac{\pi}{2}\right) \quad (34)$$

and, putting $x = x_0 + 2\pi$ (i.e. reducing to the first quadrant):

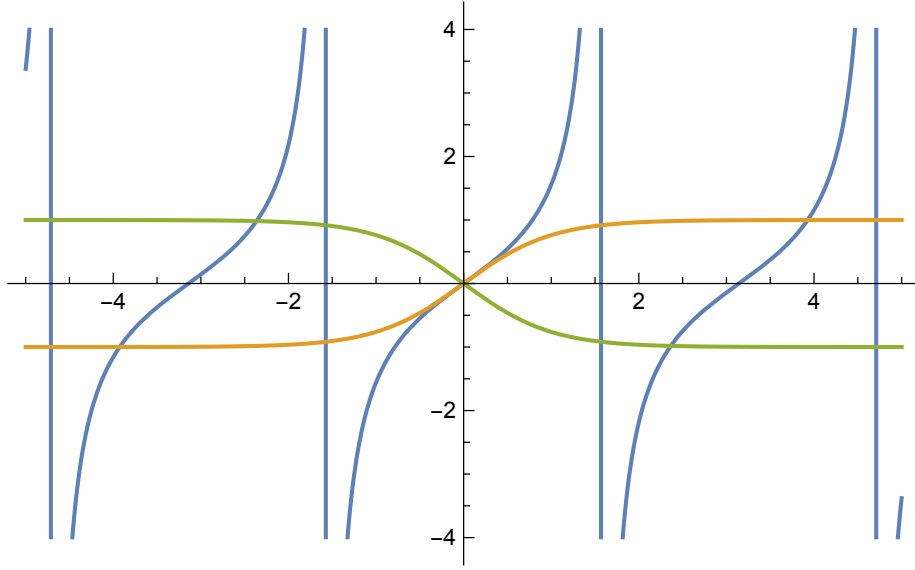


Figure 4: The plots of $\tan(x)$ (black), $\tanh(x)$ (green) and $-\tanh(x)$ (red).

$$\tan(x_0 + 2\pi) = \tanh\left(2\pi + \frac{\pi}{2}\right) \quad (35)$$

and

$$x_0 = \arctan\left(\tanh\left(\frac{5\pi}{2}\right)\right) = 0.78540$$

so:

$$x = 7.0686$$

The "exact" value is:

$$x = 7.06858 \quad (36)$$

and the relative error:

$$\varepsilon = \frac{7.06858 - 7.0686}{7.06858} = -2.83 \times 10^{-6} \quad (37)$$

So, for the practitioner working in applied physics, in a domain where the experimental error is larger than 10^{-6} , such a result is acceptable, for pragmatic reasons.

The first root of the equation

$$a \tan x + \tanh x = 0 \quad (38)$$

can be obtained for small values of a , after series expansions, as one of the roots of the equation:

$$\frac{17}{315}(a-1)x^6 + \frac{2}{15}(a+1)x^4 + \frac{1}{3}(a-1)x^2 + a+1 = 0 \quad (39)$$

For instance, if $a = 0.1$, the error of the result obtained in this way is about -4% ($x_{exact} = 1.295\,2$, $x_{approx} = 1.343\,5$).

5 The Wright omega function

The Wright omega function appears in the asymptotic form of two equations solved by Siewert and his co-workers.

In [10], the authors obtain "an exact analytical solution for the position-time relationship for an inverse-distance-squared force". Actually, they study the repulsive classical 1D movement of an electric charge in the field of another fixed charge. The repulsive force is given by the Coulomb law:

$$m \frac{d^2 r}{dt^2} = \frac{qQ}{4\pi\epsilon_0 r^2} \quad (40)$$

The initial condition is:

$$t = 0 \rightarrow \frac{dr}{dt} = 0, \quad r = r_0 \quad (41)$$

We shall define the position of the moving charge by the dimensionless function $x(t)$, defined by:

$$r(t) = r_0 x(t) \quad (42)$$

After two integrations of the equation of movement, we get the relation between position x and time t :

$$\sqrt{x(x-1)} + \ln(\sqrt{x} + \sqrt{x-1}) = \frac{t}{\tau} \quad (43)$$

with τ given by:

$$\tau = \sqrt{\frac{2\pi\epsilon_0 m r_0^3}{qQ}} \quad (44)$$

We shall study this equation at small and at large values of t . According to (42),

$$x(0) = 1 \quad (45)$$

so, for $t/\tau \ll 1$, we can put:

$$x = 1 + X, \quad X \ll 1 \quad (46)$$

and (43) can be approximated by:

$$\ln \left(1 + \frac{X}{2} + \sqrt{X} \right) = \frac{t}{\tau} - \sqrt{X} \left(1 + \frac{X}{2} \right) \quad (47)$$

and again, neglecting X with respect to \sqrt{X} :

$$\ln \left(1 + \sqrt{X} \right) = \frac{t}{\tau} - \sqrt{X} \quad (48)$$

or:

$$2\sqrt{X} = \frac{t}{\tau}$$

and finally:

$$X = \frac{1}{4\tau^2} t^2 \quad (49)$$

At very small times, $t \ll \tau$, the movement is uniformly accelerated, as expected.

Asymptotically, $x \gg 1$ and (43) gives:

$$\ln (2\sqrt{x}) = \frac{t}{\tau} - x \quad (50)$$

or:

$$2xe^{2x} = \exp \left(\frac{2t}{\tau} - \ln 2 \right) \rightarrow 2x = W \left(\exp \left(\frac{2t}{\tau} - \ln 2 \right) \right) \quad (51)$$

Consequently:

$$x = \frac{1}{2} W \left(\exp \left(\frac{2t}{\tau} - \ln 2 \right) \right) \quad (52)$$

where W is the Lambert function. In terms of the Wright omega function ω , we have the identity [37]:

$$W(e^x) = \omega(x) \quad (53)$$

So, the asymptotic formula (52) can be written equivalently as:

$$x(t) = \frac{1}{2} \omega \left(\frac{2t}{\tau} - \ln 2 \right), \quad t \gg \tau \quad (54)$$

The asymptotic expansion of the Lambert function is:

$$W(x) = \ln x - \ln(\ln x) + \frac{\ln(\ln x)}{\ln x} + \dots \quad (55)$$

Keeping only the first term, the asymptotic formula (52) gives:

$$x(t) = \frac{t}{\tau} - \frac{\ln 2}{2} \simeq \frac{t}{\tau} \quad (55)$$

This is also an intuitive result, as, at very large distances, the repulsive force produced by the fixed charge becomes negligible small, and the movement becomes almost uniform. So, the movement starts by being uniformly accelerated and ends by being uniform.

In [1], Siewert and Burniston solved the Kepler equation for hyperbolic orbits:

$$e \sinh F = F + N, \quad N > 0 \quad (56)$$

whose solution cannot be reduced to generalized Lambert function. Asymptotically, $\sinh F \rightarrow \frac{1}{2} \exp F$ and (56) becomes:

$$\frac{e}{2} \exp F = F + N, \quad N > 0 \quad (57)$$

or, with $\exp F = f$, $F = \ln f$:

$$\frac{e}{2} f = \ln f + N \quad (58)$$

so, a Wright equation, whose standard form is [37]:

$$y + \ln y = z \quad (59)$$

6 Lambert function and generalized Lambert functions

In [7], Siewert and Burniston find the solution of the equation:

$$ze^z = a \quad (60)$$

i.e. obtain an expression for the W Lambert function, a being a complex parameter [38]. In [3], Siewert solves "the familiar critical equation, described by age-diffusion theory, for a bare nuclear reactor":

$$\frac{k \exp(-B^2 \tau)}{1 + B^2 L^2} = 1 \quad (61)$$

for B^2 (the buckling).

In [9], the author solves a more complicated equation:

$$e^z \frac{z}{z+b} = a \quad (62)$$

with a - a complex parameter. So, he obtains an expression for the function $W(0; -b; a)$, which can be written, at its turn, in terms of the Mezö - Baricz function W_r . For a - real, the author refers to a paper of Wright, J. SIAM **9** (1961) 136.

6.1 Transcendental equations involving trigonometric and algebraic functions

In [6], the authors study "the critical condition for a spherical reactor, described by elementary diffusion theory, surrounded by an infinite reflector":

$$x \cot x = 1 - a - bx \quad (63)$$

We can easily obtain an approximate analytical solution of (63), using the algebraic approximation of the tangent [35], [36]. We get, in this way, instead of (63), an approximate equation:

$$x = \frac{0.45\pi (x - n\pi)}{2x - (2n - 1)\pi} (1 - a - bx) \quad (64)$$

which can be reduced to a second degree algebraic equation.

In [4], Burniston and Siewert solve the equation:

$$a \sin \zeta = \zeta \quad (65)$$

which appears in quantum mechanics (defining the eigenenergies of a particle in a square well potential), in electromagnetism, in elasticity, in optics etc. Somewhat later, Siewert obtains a simpler solution [12]. A very precise approximate analytic solution of (65) was obtained through algebraization [39]; it is useful for the calculation of energy levels in heterojunctions and quantum dots.

A more complicated variant of (65) is the Kepler equation for elliptic orbits [1] (e is the excentricity and, in this section only, has nothing to do with the basis of Nepperian logarithms):

$$e \sin E = E - M, \quad 0 < e < 1, \quad 0 < M < 2\pi \quad (66)$$

We can obtain a quite precise solution of (66) approximating the first half-bump of \sin by a cubic polynomial:

$$y(x) = ax^3 + bx^2 + x \quad (67)$$

where the coefficients a, b can be determined by imposing the conditions:

$$y\left(\frac{\pi}{2}\right) = 1, \quad y'\left(\frac{\pi}{2}\right) = 0 \quad (68)$$

We find:

$$y(x) = \frac{4}{\pi^3} (\pi - 4) x^3 - \frac{4}{\pi^2} (\pi - 3) x^2 + x \quad (69)$$

which fits quite well the function $\sin x$, for $0 < x < \pi/2$, as we can see in Fig. 5.

If $e = 0.9$, $M = 0.1$, replacing $\sin x$ with the polynom (69) in the Kepler equation (66), we obtain the solution $x_{1p} = 0.59955$, while the "exact" solution is $x_{1e} = 0.59955$; so, the error is $\varepsilon_1 \simeq 5 \times 10^{-2}$. If $e = 0.9$ and $M = 0.5$, the polynomial approximation gives $x_{2p} = 1.3821$, and the "exact" solution is $x_{2e} = 1.38441$, so the error is $\varepsilon_2 \simeq 1.7 \times 10^{-3}$. The plots in Fig. 5 show, intuitively, why the second approximation is more precise.

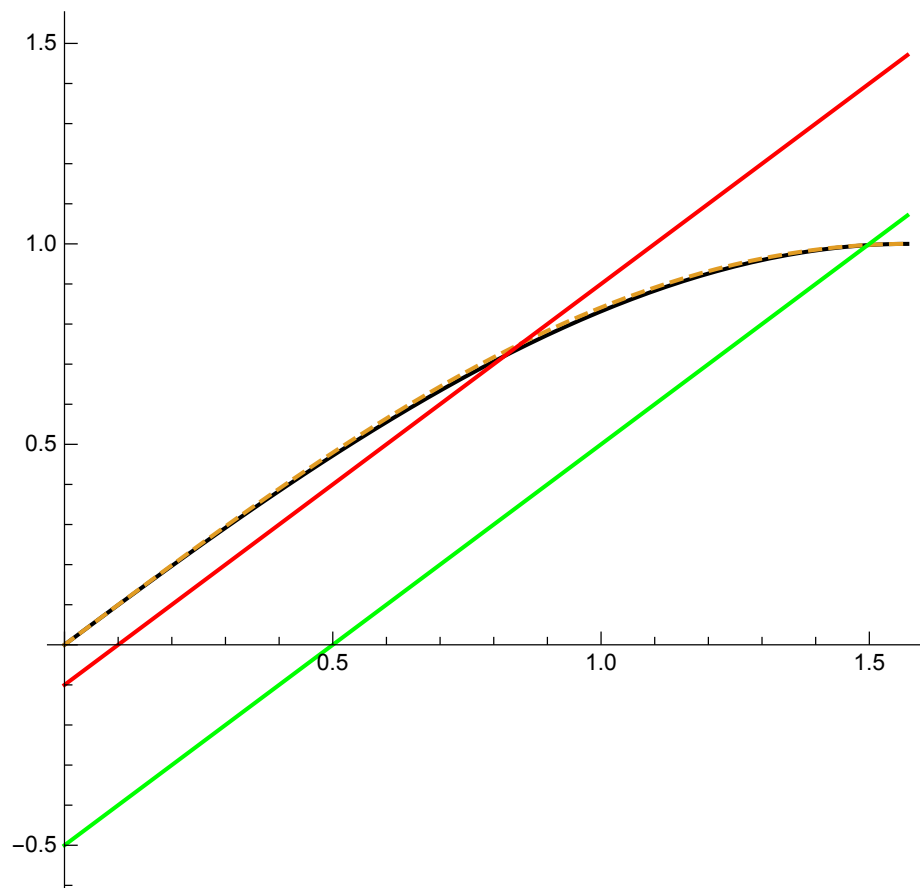


Figure 5: The plots of $y(x)$, eq. (70), black; $\sin(x)$, dashed; $x - 0.1$, red; $x - 0.5$, green.

7 Conclusions

This paper is essentially focused on the transcendental equations studied by Siewert and his coworkers, considered in conjunction with the results obtained recently in the theory of generalized Lambert functions. Siewert's exact results are compared, whenever possible, with the approximate analytical solutions of the same equations, obtained with some simple techniques. Some other results of Siewert and his coworkers, not connected to the generalized Lambert functions, are discussed; in two cases, the asymptotic behavior of Siewert's solutions are expressed in terms of Wright ω function.

As sometimes the approximate expressions of the generalized functions are very precise (and their exact expressions are difficult to obtain), these approximations could provide a useful guidance of their exact behaviour. Also, the "algebraization" of the transcendental equations (i.e. the replacement of the trigonometric functions with their various algebraic approximations) can provide, sometimes, surprisingly precise analytic approximations. They can be successfully used in applied physics or in the elementary presentation of advanced problems.

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